Latent Variable Models with Gaussian Processes

Neil D. Lawrence

GP Master Class
6th February 2017

Outline

Motivating Example

Linear Dimensionality Reduction

Non-linear Dimensionality Reduction

Outline

Motivation for Non-Linear Dimensionality Reduction

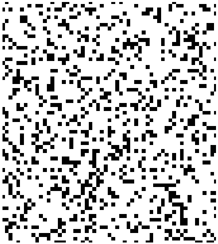
USPS Data Set Handwritten Digit

- 3648 Dimensions
  - 64 rows by 57 columns

Motivation for Non-Linear Dimensionality Reduction

USPS Data Set Handwritten Digit

- 3648 Dimensions
  - 64 rows by 57 columns
  - Space contains more than just this digit.



Motivation for Non-Linear Dimensionality Reduction

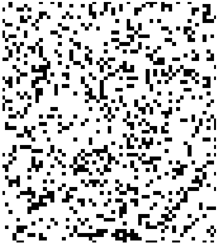
USPS Data Set Handwritten Digit

- 3648 Dimensions
  - 64 rows by 57 columns
  - Space contains more than just this digit.
  - Even if we sample every nanosecond from now until the end of the universe, you won't see the original six!

Motivation for Non-Linear Dimensionality Reduction

USPS Data Set Handwritten Digit

- 3648 Dimensions
  - 64 rows by 57 columns
  - Space contains more than just this digit.
  - Even if we sample every nanosecond from now until the end of the universe, you won't see the original six!



Simple Model of Digit

Rotate a 'Prototype'

Simple Model of Digit

Rotate a 'Prototype'

Simple Model of Digit

Rotate a 'Prototype'

Simple Model of Digit

Rotate a 'Prototype'

Simple Model of Digit

Rotate a 'Prototype'

Simple Model of Digit

Rotate a 'Prototype'

Simple Model of Digit

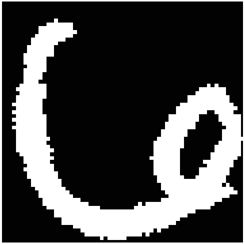
Rotate a 'Prototype'

Simple Model of Digit

Rotate a 'Prototype'

Simple Model of Digit

Rotate a 'Prototype'

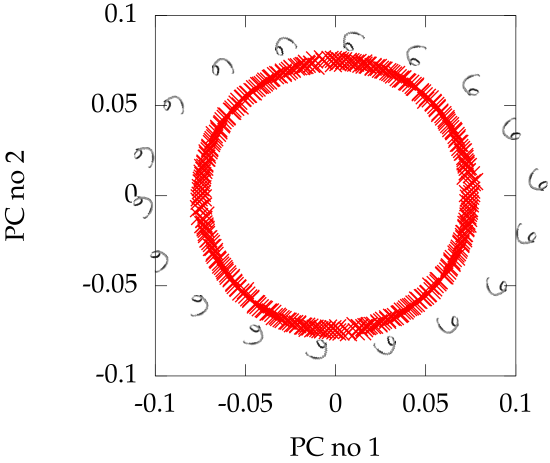


MATLAB Demo

```
demDigitsManifold([1 2], 'all')
```

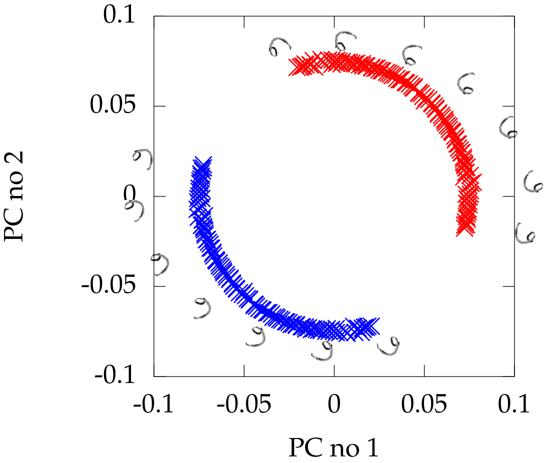
MATLAB Demo

```
demDigitsManifold([1 2], 'all')
```



MATLAB Demo

```
demDigitsManifold([1 2], 'sixnine')
```



Low Dimensional Manifolds

Pure Rotation is too Simple

- In practice the data may undergo several distortions.
  - *e.g.* digits undergo 'thinning', translation and rotation.
- For data with 'structure':
  - we expect fewer distortions than dimensions;
  - we therefore expect the data to live on a lower dimensional manifold.
- Conclusion: deal with high dimensional data by looking for lower dimensional non-linear embedding.

Outline

Notation

q— dimension of latent/embedded space
p— dimension of data space
n— number of data points

$$\text{data, } \mathbf{Y} = [\mathbf{y}_{1,:}, \ldots, \mathbf{y}_{n,:}]^\top = \left[\mathbf{y}_{:,1}, \ldots, \mathbf{y}_{:,p}\right] \in \Re^{n \times p}$$

$$\textit{centred} \text{ data, } \hat{\mathbf{Y}} = [\hat{\mathbf{y}}_{1,:}, \ldots, \hat{\mathbf{y}}_{n,:}]^\top = \left[\hat{\mathbf{y}}_{:,1}, \ldots, \hat{\mathbf{y}}_{:,p}\right] \in \Re^{n \times p},$$

$$\hat{\mathbf{y}}_{i,:} = \mathbf{y}_{i,:} - \boldsymbol{\mu}$$

$$\text{latent variables, } \mathbf{X} = [\mathbf{x}_{1,:}, \ldots, \mathbf{x}_{n,:}]^\top = \left[\mathbf{x}_{:,1}, \ldots, \mathbf{x}_{:,q}\right] \in \Re^{n \times q}$$

$$\text{mapping matrix, } \mathbf{W} \in \Re^{p \times q}$$

$\mathbf{a}_{i,:}$ is a vector from the ith row of a given matrix $\mathbf{A}$
$\mathbf{a}_{:,j}$ is a vector from the jth row of a given matrix $\mathbf{A}$

Reading Notation

$\mathbf{X}$ and $\mathbf{Y}$ are *design matrices*

- Data covariance given by $\frac{1}{n}\hat{\mathbf{Y}}^\top\hat{\mathbf{Y}}$

$$\text{cov}\left(\mathbf{Y}\right) = \frac{1}{n}\sum_{i=1}^{n}\hat{\mathbf{y}}_{i,:}\hat{\mathbf{y}}_{i,:}^\top = \frac{1}{n}\hat{\mathbf{Y}}^\top\hat{\mathbf{Y}} = \mathbf{S}.$$

- Inner product matrix given by $\mathbf{Y}\mathbf{Y}^\top$

$$\mathbf{K} = \left(k_{i,j}\right)_{i,j}, \qquad k_{i,j} = \mathbf{y}_{i,:}^\top\mathbf{y}_{j,:}$$

Linear Dimensionality Reduction

- Find a lower dimensional plane embedded in a higher dimensional space.
- The plane is described by the matrix $\mathbf{W} \in \Re^{p \times q}$.

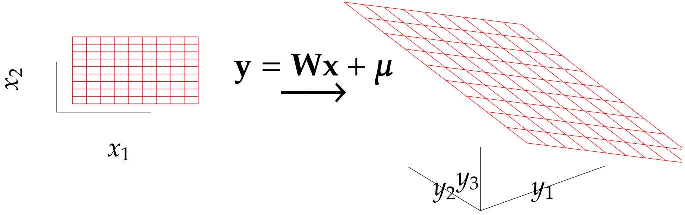


Figure: Mapping a two dimensional plane to a higher dimensional space in a linear way. Data are generated by corrupting points on the plane with noise.

Linear Dimensionality Reduction

Linear Latent Variable Model

- Represent data, $\mathbf{Y}$, with a lower dimensional set of latent variables $\mathbf{X}$.
- Assume a linear relationship of the form

$$\mathbf{y}_{i,:} = \mathbf{W}\mathbf{x}_{i,:} + \boldsymbol{\epsilon}_{i,:},$$

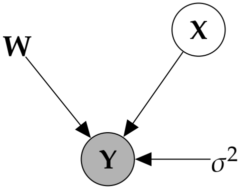
where

$$\boldsymbol{\epsilon}_{i,:} \sim \mathcal{N}\left(\mathbf{0}, \sigma^2\mathbf{I}\right).$$

Linear Latent Variable Model

Probabilistic PCA

- Define *linear-Gaussian relationship* between latent variables and data.

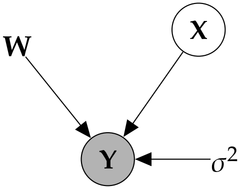


$$p\left(\mathbf{Y}|\mathbf{X},\mathbf{W}\right)=\prod_{i=1}^{n}\mathcal{N}\left(\mathbf{y}_{i,:}|\mathbf{W}\mathbf{x}_{i,:},\sigma^2\mathbf{I}\right)$$

Linear Latent Variable Model

Probabilistic PCA

- Define *linear-Gaussian relationship* between latent variables and data.
- **Standard** Latent variable approach:

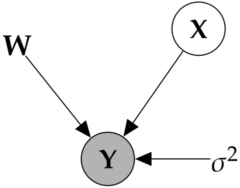


$$p(\mathbf{Y}|\mathbf{X},\mathbf{W}) = \prod_{i=1}^{n} \mathcal{N}\left(\mathbf{y}_{i,:}|\mathbf{W}\mathbf{x}_{i,:},\sigma^2\mathbf{I}\right)$$

Linear Latent Variable Model

Probabilistic PCA

- Define *linear-Gaussian relationship* between latent variables and data.
- **Standard** Latent variable approach:
  - Define Gaussian prior over *latent space*, **X**.



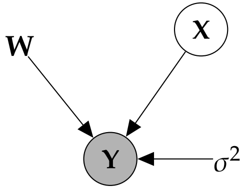
$$p(\mathbf{Y}|\mathbf{X},\mathbf{W}) = \prod_{i=1}^{n} \mathcal{N}\left(\mathbf{y}_{i,:}|\mathbf{W}\mathbf{x}_{i,:},\sigma^2\mathbf{I}\right)$$

$$p(\mathbf{X}) = \prod_{i=1}^{n} \mathcal{N}\left(\mathbf{x}_{i,:}|\mathbf{0},\mathbf{I}\right)$$

Linear Latent Variable Model

Probabilistic PCA

- Define *linear-Gaussian relationship* between latent variables and data.
- **Standard** Latent variable approach:
  - Define Gaussian prior over *latent space*, **X**.
  - Integrate out *latent variables*.



$$p(\mathbf{Y}|\mathbf{X},\mathbf{W}) = \prod_{i=1}^{n} \mathcal{N}\left(\mathbf{y}_{i,:}|\mathbf{W}\mathbf{x}_{i,:}, \sigma^2\mathbf{I}\right)$$

$$p(\mathbf{X}) = \prod_{i=1}^{n} \mathcal{N}\left(\mathbf{x}_{i,:}|\mathbf{0},\mathbf{I}\right)$$

$$p(\mathbf{Y}|\mathbf{W}) = \prod_{i=1}^{n} \mathcal{N}\left(\mathbf{y}_{i,:}|\mathbf{0},\mathbf{W}\mathbf{W}^\top + \sigma^2\mathbf{I}\right)$$

Computation of the Marginal Likelihood

$$\mathbf{y}_{i,:} = \mathbf{W}\mathbf{x}_{i,:} + \boldsymbol{\epsilon}_{i,:}, \quad \mathbf{x}_{i,:} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{I}\right), \quad \boldsymbol{\epsilon}_{i,:} \sim \mathcal{N}\left(\mathbf{0}, \sigma^2\mathbf{I}\right)$$

Computation of the Marginal Likelihood

$$\mathbf{y}_{i,:} = \mathbf{W}\mathbf{x}_{i,:} + \boldsymbol{\epsilon}_{i,:}, \quad \mathbf{x}_{i,:} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{I}\right), \quad \boldsymbol{\epsilon}_{i,:} \sim \mathcal{N}\left(\mathbf{0}, \sigma^2\mathbf{I}\right)$$

$$\mathbf{W}\mathbf{x}_{i,:} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{W}\mathbf{W}^\top\right),$$

Computation of the Marginal Likelihood

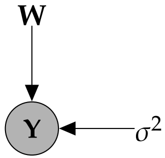
$$\mathbf{y}_{i,:} = \mathbf{W}\mathbf{x}_{i,:} + \boldsymbol{\epsilon}_{i,:}, \quad \mathbf{x}_{i,:} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}), \quad \boldsymbol{\epsilon}_{i,:} \sim \mathcal{N}\left(\mathbf{0}, \sigma^2\mathbf{I}\right)$$

$$\mathbf{W}\mathbf{x}_{i,:} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{W}\mathbf{W}^\top\right),$$

$$\mathbf{W}\mathbf{x}_{i,:} + \boldsymbol{\epsilon}_{i,:} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{W}\mathbf{W}^\top + \sigma^2\mathbf{I}\right)$$

Linear Latent Variable Model II

Probabilistic PCA Max. Likelihood Soln (Tipping and Bishop, 1999)



$$p\left(\mathbf{Y}|\mathbf{W}\right) = \prod_{i=1}^{n} \mathcal{N}\left(\mathbf{y}_{i,:}|\mathbf{0}, \mathbf{W}\mathbf{W}^{\top} + \sigma^2\mathbf{I}\right)$$

Linear Latent Variable Model II

Probabilistic PCA Max. Likelihood Soln (Tipping and Bishop, 1999)

$$p(\mathbf{Y}|\mathbf{W}) = \prod_{i=1}^{n} \mathcal{N}(\mathbf{y}_{i,:}|\mathbf{0}, \mathbf{C}), \quad \mathbf{C} = \mathbf{W}\mathbf{W}^{\top} + \sigma^2\mathbf{I}$$

Linear Latent Variable Model II

Probabilistic PCA Max. Likelihood Soln (Tipping and Bishop, 1999)

$$p(\mathbf{Y}|\mathbf{W}) = \prod_{i=1}^{n} \mathcal{N}(\mathbf{y}_{i,:}|\mathbf{0}, \mathbf{C}), \quad \mathbf{C} = \mathbf{W}\mathbf{W}^{\top} + \sigma^2\mathbf{I}$$

$$\log p(\mathbf{Y}|\mathbf{W}) = -\frac{n}{2}\log|\mathbf{C}| - \frac{1}{2}\mathrm{tr}\left(\mathbf{C}^{-1}\mathbf{Y}^{\top}\mathbf{Y}\right) + \mathrm{const.}$$

Linear Latent Variable Model II

Probabilistic PCA Max. Likelihood Soln (Tipping and Bishop, 1999)

$$p(\mathbf{Y}|\mathbf{W}) = \prod_{i=1}^{n} \mathcal{N}\left(\mathbf{y}_{i,:}|\mathbf{0}, \mathbf{C}\right), \quad \mathbf{C} = \mathbf{W}\mathbf{W}^{\top} + \sigma^2\mathbf{I}$$

$$\log p(\mathbf{Y}|\mathbf{W}) = -\frac{n}{2}\log|\mathbf{C}| - \frac{1}{2}\mathrm{tr}\left(\mathbf{C}^{-1}\mathbf{Y}^{\top}\mathbf{Y}\right) + \text{const.}$$

If $\mathbf{U}_q$ are first q principal eigenvectors of $n^{-1}\mathbf{Y}^{\top}\mathbf{Y}$ and the corresponding eigenvalues are $\mathbf{\Lambda}_q$,

Linear Latent Variable Model II

Probabilistic PCA Max. Likelihood Soln (Tipping and Bishop, 1999)

$$p(\mathbf{Y}|\mathbf{W}) = \prod_{i=1}^{n} \mathcal{N}\left(\mathbf{y}_{i,:}|\mathbf{0}, \mathbf{C}\right), \quad \mathbf{C} = \mathbf{W}\mathbf{W}^{\top} + \sigma^2\mathbf{I}$$

$$\log p(\mathbf{Y}|\mathbf{W}) = -\frac{n}{2}\log|\mathbf{C}| - \frac{1}{2}\mathrm{tr}\left(\mathbf{C}^{-1}\mathbf{Y}^{\top}\mathbf{Y}\right) + \text{const.}$$

If $\mathbf{U}_q$ are first q principal eigenvectors of $n^{-1}\mathbf{Y}^{\top}\mathbf{Y}$ and the corresponding eigenvalues are $\mathbf{\Lambda}_q$,

$$\mathbf{W} = \mathbf{U}_q\mathbf{L}\mathbf{R}^{\top}, \quad \mathbf{L} = \left(\mathbf{\Lambda}_q - \sigma^2\mathbf{I}\right)^{\frac{1}{2}}$$

where $\mathbf{R}$ is an arbitrary rotation matrix.

Outline

Difficulty for Probabilistic Approaches

- Propagate a probability distribution through a non-linear mapping.
- Normalisation of distribution becomes intractable.

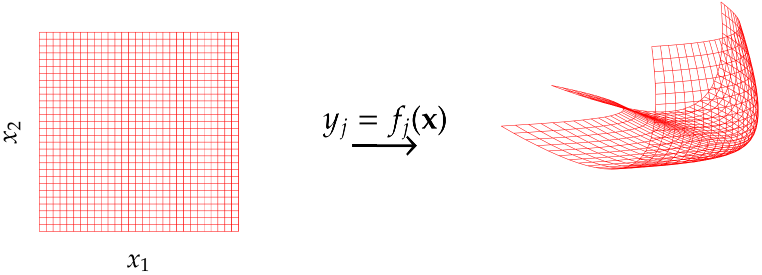


Figure: A three dimensional manifold formed by mapping from a two dimensional space to a three dimensional space.

Difficulty for Probabilistic Approaches

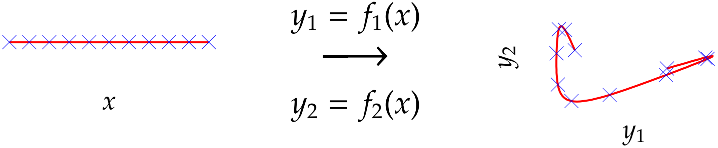


Figure: A string in two dimensions, formed by mapping from one dimension, x, line to a two dimensional space, $[y_1, \; y_2]$ using nonlinear functions $f_1(\cdot)$ and $f_2(\cdot)$.

Difficulty for Probabilistic Approaches

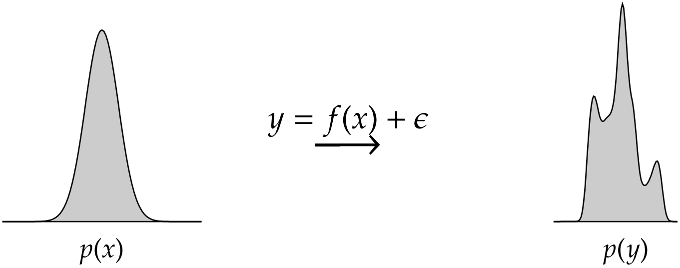
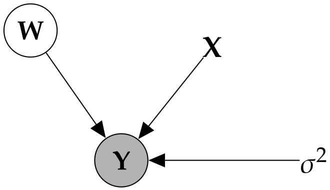


Figure: A Gaussian distribution propagated through a non-linear mapping. $y_i = f(x_i) + \epsilon_i$. $\epsilon \sim \mathcal{N}\left(0, 0.2^2\right)$ and $f(\cdot)$ uses RBF basis, 100 centres between -4 and 4 and $\ell = 0.1$. New distribution over y (right) is multimodal and difficult to normalize.

Linear Latent Variable Model III

Dual Probabilistic PCA

- Define *linear-Gaussian relationship* between latent variables and data.

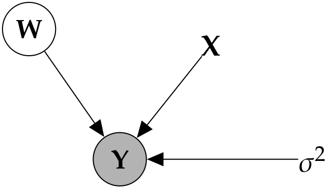


$$p(\mathbf{Y}|\mathbf{X},\mathbf{W}) = \prod_{i=1}^{n} \mathcal{N}\left(\mathbf{y}_{i,:}|\mathbf{W}\mathbf{x}_{i,:},\sigma^2\mathbf{I}\right)$$

Linear Latent Variable Model III

Dual Probabilistic PCA

- Define *linear-Gaussian relationship* between latent variables and data.
- **Novel** Latent variable approach:

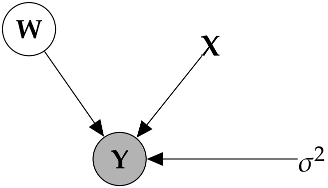


$$p(\mathbf{Y}|\mathbf{X}, \mathbf{W}) = \prod_{i=1}^{n} \mathcal{N}\left(\mathbf{y}_{i,:}|\mathbf{W}\mathbf{x}_{i,:}, \sigma^2\mathbf{I}\right)$$

Linear Latent Variable Model III

Dual Probabilistic PCA

- Define *linear-Gaussian relationship* between latent variables and data.
- **Novel** Latent variable approach:
    - Define Gaussian prior over *parameters*, **W**.



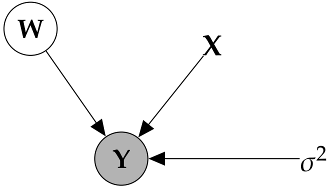
$$p\left(\mathbf{Y}|\mathbf{X},\mathbf{W}\right) = \prod_{i=1}^{n} \mathcal{N}\left(\mathbf{y}_{i,:}|\mathbf{W}\mathbf{x}_{i,:},\sigma^2\mathbf{I}\right)$$

$$p\left(\mathbf{W}\right) = \prod_{i=1}^{p} \mathcal{N}\left(\mathbf{w}_{i,:}|\mathbf{0},\mathbf{I}\right)$$

Linear Latent Variable Model III

Dual Probabilistic PCA

- Define *linear-Gaussian relationship* between latent variables and data.
- **Novel** Latent variable approach:
  - Define Gaussian prior over *parameters*, **W**.
  - Integrate out *parameters*.



$$p\left(\mathbf{Y}|\mathbf{X},\mathbf{W}\right)=\prod_{i=1}^{n}\mathcal{N}\left(\mathbf{y}_{i,:}|\mathbf{W}\mathbf{x}_{i,:},\sigma^{2}\mathbf{I}\right)$$

$$p\left(\mathbf{W}\right)=\prod_{i=1}^{p}\mathcal{N}\left(\mathbf{w}_{i,:}|\mathbf{0},\mathbf{I}\right)$$

$$p\left(\mathbf{Y}|\mathbf{X}\right)=\prod_{j=1}^{p}\mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0},\mathbf{X}\mathbf{X}^{\top}+\sigma^{2}\mathbf{I}\right)$$

Computation of the Marginal Likelihood

$$\mathbf{y}_{:,j} = \mathbf{X}\mathbf{w}_{:,j} + \boldsymbol{\epsilon}_{:,j}, \quad \mathbf{w}_{:,j} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{I}\right), \quad \boldsymbol{\epsilon}_{i,:} \sim \mathcal{N}\left(\mathbf{0}, \sigma^2\mathbf{I}\right)$$

Computation of the Marginal Likelihood

$$\mathbf{y}_{:,j} = \mathbf{X}\mathbf{w}_{:,j} + \boldsymbol{\epsilon}_{:,j}, \quad \mathbf{w}_{:,j} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{I}\right), \quad \boldsymbol{\epsilon}_{i,:} \sim \mathcal{N}\left(\mathbf{0}, \sigma^2\mathbf{I}\right)$$

$$\mathbf{X}\mathbf{w}_{:,j} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{X}\mathbf{X}^\top\right),$$

Computation of the Marginal Likelihood

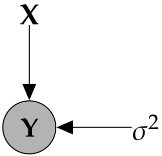
$$\mathbf{y}_{:,j} = \mathbf{X}\mathbf{w}_{:,j} + \boldsymbol{\epsilon}_{:,j}, \quad \mathbf{w}_{:,j} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}), \quad \boldsymbol{\epsilon}_{i,:} \sim \mathcal{N}\left(\mathbf{0}, \sigma^2\mathbf{I}\right)$$

$$\mathbf{X}\mathbf{w}_{:,j} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{X}\mathbf{X}^\top\right),$$

$$\mathbf{X}\mathbf{w}_{:,j} + \boldsymbol{\epsilon}_{:,j} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{X}\mathbf{X}^\top + \sigma^2\mathbf{I}\right)$$

Linear Latent Variable Model IV

Dual Probabilistic PCA Max. Likelihood Soln (Lawrence, 2004, 2005)



$$p\left(\mathbf{Y}|\mathbf{X}\right) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{X}\mathbf{X}^{\top} + \sigma^2\mathbf{I}\right)$$

Linear Latent Variable Model IV

Dual PPCA Max. Likelihood Soln (Lawrence, 2004, 2005)

$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right), \quad \mathbf{K} = \mathbf{X}\mathbf{X}^{\top} + \sigma^2\mathbf{I}$$

Linear Latent Variable Model IV

PPCA Max. Likelihood Soln (Tipping and Bishop, 1999)

$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right), \quad \mathbf{K} = \mathbf{X}\mathbf{X}^{\top} + \sigma^2\mathbf{I}$$

$$\log p(\mathbf{Y}|\mathbf{X}) = -\frac{p}{2}\log|\mathbf{K}| - \frac{1}{2}\mathrm{tr}\left(\mathbf{K}^{-1}\mathbf{Y}\mathbf{Y}^{\top}\right) + \text{const.}$$

Linear Latent Variable Model IV

PPCA Max. Likelihood Soln

$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right), \quad \mathbf{K} = \mathbf{X}\mathbf{X}^{\top} + \sigma^2\mathbf{I}$$

$$\log p(\mathbf{Y}|\mathbf{X}) = -\frac{p}{2}\log|\mathbf{K}| - \frac{1}{2}\mathrm{tr}\left(\mathbf{K}^{-1}\mathbf{Y}\mathbf{Y}^{\top}\right) + \text{const.}$$

If $\mathbf{U}_q'$ are first q principal eigenvectors of $p^{-1}\mathbf{Y}\mathbf{Y}^{\top}$ and the corresponding eigenvalues are $\mathbf{\Lambda}_q$,

Linear Latent Variable Model IV

PPCA Max. Likelihood Soln

$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right), \quad \mathbf{K} = \mathbf{X}\mathbf{X}^{\top} + \sigma^2\mathbf{I}$$

$$\log p(\mathbf{Y}|\mathbf{X}) = -\frac{p}{2}\log|\mathbf{K}| - \frac{1}{2}\mathrm{tr}\left(\mathbf{K}^{-1}\mathbf{Y}\mathbf{Y}^{\top}\right) + \text{const.}$$

If $\mathbf{U}_q'$ are first q principal eigenvectors of $p^{-1}\mathbf{Y}\mathbf{Y}^{\top}$ and the corresponding eigenvalues are $\mathbf{\Lambda}_q$,

$$\mathbf{X} = \mathbf{U}_q'\mathbf{L}\mathbf{R}^{\top}, \quad \mathbf{L} = \left(\mathbf{\Lambda}_q - \sigma^2\mathbf{I}\right)^{\frac{1}{2}}$$

where $\mathbf{R}$ is an arbitrary rotation matrix.

Linear Latent Variable Model IV

Dual PPCA Max. Likelihood Soln (Lawrence, 2004, 2005)

$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right), \quad \mathbf{K} = \mathbf{X}\mathbf{X}^{\top} + \sigma^2\mathbf{I}$$

$$\log p(\mathbf{Y}|\mathbf{X}) = -\frac{p}{2}\log|\mathbf{K}| - \frac{1}{2}\mathrm{tr}\left(\mathbf{K}^{-1}\mathbf{Y}\mathbf{Y}^{\top}\right) + \text{const.}$$

If $\mathbf{U}_q'$ are first q principal eigenvectors of $p^{-1}\mathbf{Y}\mathbf{Y}^{\top}$ and the corresponding eigenvalues are $\mathbf{\Lambda}_q$,

$$\mathbf{X} = \mathbf{U}_q'\mathbf{L}\mathbf{R}^{\top}, \quad \mathbf{L} = \left(\mathbf{\Lambda}_q - \sigma^2\mathbf{I}\right)^{\frac{1}{2}}$$

where $\mathbf{R}$ is an arbitrary rotation matrix.

Linear Latent Variable Model IV

PPCA Max. Likelihood Soln (Tipping and Bishop, 1999)

$$p(\mathbf{Y}|\mathbf{W}) = \prod_{i=1}^{n} \mathcal{N}(\mathbf{y}_{i,:}|\mathbf{0}, \mathbf{C}), \quad \mathbf{C} = \mathbf{W}\mathbf{W}^{\top} + \sigma^2\mathbf{I}$$

$$\log p(\mathbf{Y}|\mathbf{W}) = -\frac{n}{2}\log|\mathbf{C}| - \frac{1}{2}\mathrm{tr}\left(\mathbf{C}^{-1}\mathbf{Y}^{\top}\mathbf{Y}\right) + \text{const.}$$

If $\mathbf{U}_q$ are first q principal eigenvectors of $n^{-1}\mathbf{Y}^{\top}\mathbf{Y}$ and the corresponding eigenvalues are $\mathbf{\Lambda}_q$,

$$\mathbf{W} = \mathbf{U}_q\mathbf{L}\mathbf{R}^{\top}, \quad \mathbf{L} = \left(\mathbf{\Lambda}_q - \sigma^2\mathbf{I}\right)^{\frac{1}{2}}$$

where $\mathbf{R}$ is an arbitrary rotation matrix.

Equivalence of Formulations

The Eigenvalue Problems are equivalent

- Solution for Probabilistic PCA (solves for the mapping)

$$\mathbf{Y}^{\top}\mathbf{Y}\mathbf{U}_q = \mathbf{U}_q\mathbf{\Lambda}_q \qquad \mathbf{W} = \mathbf{U}_q\mathbf{L}\mathbf{R}^{\top}$$

- Solution for Dual Probabilistic PCA (solves for the latent positions)

$$\mathbf{Y}\mathbf{Y}^{\top}\mathbf{U}_q' = \mathbf{U}_q'\mathbf{\Lambda}_q \qquad \mathbf{X} = \mathbf{U}_q'\mathbf{L}\mathbf{R}^{\top}$$

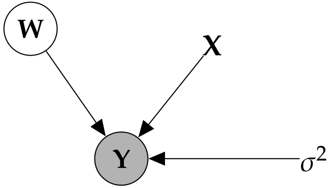
- Equivalence is from

$$\mathbf{U}_q = \mathbf{Y}^{\top}\mathbf{U}_q'\mathbf{\Lambda}_q^{-\frac{1}{2}}$$

Non-Linear Latent Variable Model

Dual Probabilistic PCA

- Define *linear-Gaussian relationship* between latent variables and data.
- **Novel** Latent variable approach:
  - Define Gaussian prior over *parameteters*, **W**.
  - Integrate out *parameters*.



$$p(\mathbf{Y}|\mathbf{X},\mathbf{W}) = \prod_{i=1}^{n} \mathcal{N}\left(\mathbf{y}_{i,:}|\mathbf{W}\mathbf{x}_{i,:}, \sigma^2\mathbf{I}\right)$$

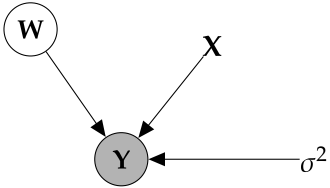
$$p(\mathbf{W}) = \prod_{i=1}^{p} \mathcal{N}\left(\mathbf{w}_{i,:}|\mathbf{0},\mathbf{I}\right)$$

$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0},\mathbf{X}\mathbf{X}^\top + \sigma^2\mathbf{I}\right)$$

Non-Linear Latent Variable Model

Dual Probabilistic PCA

- Inspection of the marginal likelihood shows ...

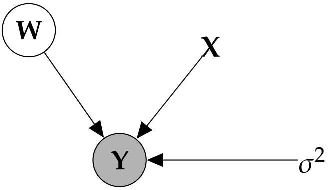


$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{X}\mathbf{X}^{\top} + \sigma^2\mathbf{I}\right)$$

Non-Linear Latent Variable Model

Dual Probabilistic PCA

- Inspection of the marginal likelihood shows ...
  - The covariance matrix is a covariance function.



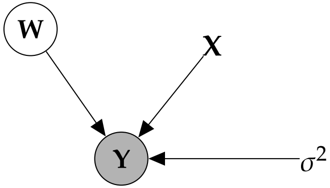
$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right)$$

$$\mathbf{K} = \mathbf{X}\mathbf{X}^\top + \sigma^2\mathbf{I}$$

Non-Linear Latent Variable Model

Dual Probabilistic PCA

- Inspection of the marginal likelihood shows ...
  - The covariance matrix is a covariance function.
  - We recognise it as the 'linear kernel'.



$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right)$$

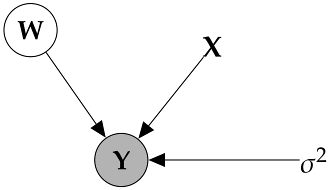
$$\mathbf{K} = \mathbf{X}\mathbf{X}^{\top} + \sigma^2\mathbf{I}$$

This is a product of Gaussian processes with linear kernels.

Non-Linear Latent Variable Model

Dual Probabilistic PCA

- Inspection of the marginal likelihood shows ...
  - The covariance matrix is a covariance function.
  - We recognise it as the 'linear kernel'.
  - We call this the Gaussian Process Latent Variable model (GP-LVM).



$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right)$$

$$\mathbf{K} = ?$$

Replace linear kernel with non-linear kernel for non-linear model.

Exponentiated Quadratic (EQ) Covariance

- The EQ covariance has the form $k_{i,j} = k\left(\mathbf{x}_{i,:}, \mathbf{x}_{j,:}\right)$, where

$$k\left(\mathbf{x}_{i,:}, \mathbf{x}_{j,:}\right) = \alpha \exp\left(-\frac{\left\|\mathbf{x}_{i,:} - \mathbf{x}_{j,:}\right\|_2^2}{2\ell^2}\right).$$

- No longer possible to optimise wrt $\mathbf{X}$ via an eigenvalue problem.
- Instead find gradients with respect to $\mathbf{X}, \alpha, \ell$ and σ^2 and optimise using conjugate gradients.

Applications

Style Based Inverse Kinematics

- Facilitating animation through modeling human motion (Grochow et al., 2004)

Tracking

- Tracking using human motion models (Urtasun et al., 2005, 2006)

Assisted Animation

- Generalizing drawings for animation (Baxter and Anjyo, 2006)

Shape Models

- Inferring shape (e.g. pose from silhouette). (Ek et al., 2008b,a; Priacuriu and Reid, 2011a,b)

Stick Man

Generalization with less Data than Dimensions

- Powerful uncertainly handling of GPs leads to surprising properties.
- Non-linear models can be used where there are fewer data points than dimensions *without overfitting*.
- Example: Modelling a stick man in 102 dimensions with 55 data points!

Stick Man II

`demStick1`

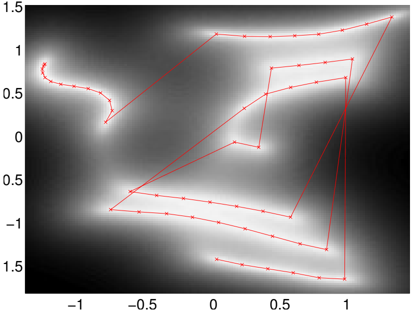


Figure: The latent space for the stick man motion capture data.

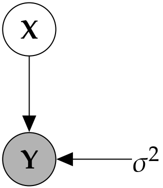
Selecting Data Dimensionality

- GP-LVM Provides probabilistic non-linear dimensionality reduction.
- How to select the dimensionality?
- Need to estimate marginal likelihood.
- In standard GP-LVM it increases with increasing q.

Integrate Mapping Function and Latent Variables

Bayesian GP-LVM

- Start with a standard GP-LVM.

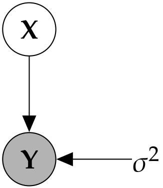


$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right)$$

Integrate Mapping Function and Latent Variables

Bayesian GP-LVM

- Start with a standard GP-LVM.
- Apply standard latent variable approach:
  - Define Gaussian prior over *latent space*, $\mathbf{X}$.

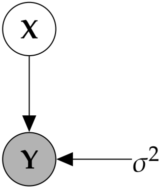


$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right)$$

Integrate Mapping Function and Latent Variables

Bayesian GP-LVM

- Start with a standard GP-LVM.
- Apply standard latent variable approach:
  - Define Gaussian prior over *latent space*, **X**.
  - Integrate out *latent variables*.



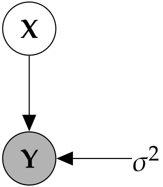
$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right)$$

$$p(\mathbf{X}) = \prod_{j=1}^{q} \mathcal{N}\left(\mathbf{x}_{:,j}|\mathbf{0}, \alpha_i^{-2}\mathbf{I}\right)$$

Integrate Mapping Function and Latent Variables

Bayesian GP-LVM

- Start with a standard GP-LVM.
- Apply standard latent variable approach:
  - Define Gaussian prior over *latent space*, $\mathbf{X}$.
  - Integrate out *latent variables*.
  - Unfortunately integration is intractable.



$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right)$$

$$p(\mathbf{X}) = \prod_{j=1}^{q} \mathcal{N}\left(\mathbf{x}_{:,j}|\mathbf{0}, \alpha_i^{-2}\mathbf{I}\right)$$

$$p(\mathbf{Y}|\boldsymbol{\alpha}) = ??$$

Standard Variational Approach Fails

- Standard variational bound has the form:

$$\mathcal{L} = \left\langle \log p(\mathbf{y}|\mathbf{X}) \right\rangle_{q(\mathbf{X})} + \mathrm{KL}\left(q(\mathbf{X}) \,\|\, p(\mathbf{X})\right)$$

Standard Variational Approach Fails

- Standard variational bound has the form:

$$\mathcal{L} = \langle \log p(\mathbf{y}|\mathbf{X}) \rangle_{q(\mathbf{X})} + \mathrm{KL}\left(q(\mathbf{X}) \,\|\, p(\mathbf{X})\right)$$

- Requires expectation of $\log p(\mathbf{y}|\mathbf{X})$ under $q(\mathbf{X})$.

$$\log p(\mathbf{y}|\mathbf{X}) = -\frac{1}{2}\mathbf{y}^\top \left(\mathbf{K}_{\mathbf{f},\mathbf{f}} + \sigma^2\mathbf{I}\right)^{-1} \mathbf{y} - \frac{1}{2}\log\left|\mathbf{K}_{\mathbf{f},\mathbf{f}} + \sigma^2\mathbf{I}\right| - \frac{n}{2}\log 2\pi$$

Standard Variational Approach Fails

- Standard variational bound has the form:

$$\mathcal{L} = \left\langle \log p(\mathbf{y}|\mathbf{X}) \right\rangle_{q(\mathbf{X})} + \mathrm{KL}\left(q(\mathbf{X}) \,\|\, p(\mathbf{X})\right)$$

- Requires expectation of $\log p(\mathbf{y}|\mathbf{X})$ under $q(\mathbf{X})$.

$$\log p(\mathbf{y}|\mathbf{X}) = -\frac{1}{2}\mathbf{y}^{\top}\left(\mathbf{K}_{\mathbf{f},\mathbf{f}} + \sigma^2\mathbf{I}\right)^{-1}\mathbf{y} - \frac{1}{2}\log\left|\mathbf{K}_{\mathbf{f},\mathbf{f}} + \sigma^2\mathbf{I}\right| - \frac{n}{2}\log 2\pi$$

- Extremely difficult to compute because $\mathbf{K}_{\mathbf{f},\mathbf{f}}$ is dependent on $\mathbf{X}$ and appears in the inverse.

Variational Bayesian GP-LVM

- Consider collapsed variational bound,

$$p(\mathbf{y}) \geq \prod_{i=1}^{n} c_i \int \mathcal{N}\left(\mathbf{y}|\left\langle \mathbf{f} \right\rangle, \sigma^2\mathbf{I}\right) p(\mathbf{u}) \mathrm{d}\mathbf{u}$$

Variational Bayesian GP-LVM

- Consider collapsed variational bound,

$$p(\mathbf{y}|\mathbf{X}) \geq \prod_{i=1}^{n} c_i \int \mathcal{N}\left(\mathbf{y}|\left\langle \mathbf{f} \right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right) p(\mathbf{u}) \mathrm{d}\mathbf{u}$$

Variational Bayesian GP-LVM

- Consider collapsed variational bound,

$$\int p(\mathbf{y}|\mathbf{X})p(\mathbf{X})\mathrm{d}\mathbf{X} \geq \int \prod_{i=1}^{n} c_i \mathcal{N}\left(\mathbf{y}|\left\langle \mathbf{f}\right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right) p(\mathbf{X})\mathrm{d}\mathbf{X}p(\mathbf{u})\mathrm{d}\mathbf{u}$$

Variational Bayesian GP-LVM

- Consider collapsed variational bound,

$$\int p(\mathbf{y}|\mathbf{X})p(\mathbf{X})\mathrm{d}\mathbf{X} \geq \int \prod_{i=1}^{n} c_i \mathcal{N}\left(\mathbf{y}|\left\langle\mathbf{f}\right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right) p(\mathbf{X})\mathrm{d}\mathbf{X}p(\mathbf{u})\mathrm{d}\mathbf{u}$$

- Apply variational lower bound to the inner integral.

Variational Bayesian GP-LVM

- Consider collapsed variational bound,

$$\int p(\mathbf{y}|\mathbf{X})p(\mathbf{X})\mathrm{d}\mathbf{X} \geq \int \prod_{i=1}^{n} c_i \mathcal{N}\left(\mathbf{y}|\left\langle\mathbf{f}\right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right) p(\mathbf{X})\mathrm{d}\mathbf{X} p(\mathbf{u})\mathrm{d}\mathbf{u}$$

- Apply variational lower bound to the inner integral.

$$\int \prod_{i=1}^{n} c_i \mathcal{N}\left(\mathbf{y}|\left\langle\mathbf{f}\right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right) p(\mathbf{X})\mathrm{d}\mathbf{X}$$

$$\geq \left\langle \sum_{i=1}^{n} \log c_i \right\rangle_{q(\mathbf{X})}$$

$$+ \left\langle \log \mathcal{N}\left(\mathbf{y}|\left\langle\mathbf{f}\right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right)\right\rangle_{q(\mathbf{X})}$$

$$+ \mathrm{KL}\left(q(\mathbf{X}) \,\|\, p(\mathbf{X})\right)$$

Variational Bayesian GP-LVM

- Consider collapsed variational bound,

$$\int p(\mathbf{y}|\mathbf{X})p(\mathbf{X})\mathrm{d}\mathbf{X} \geq \int \prod_{i=1}^{n} c_i \mathcal{N}\left(\mathbf{y}|\left\langle \mathbf{f}\right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right) p(\mathbf{X})\mathrm{d}\mathbf{X}p(\mathbf{u})\mathrm{d}\mathbf{u}$$

- Apply variational lower bound to the inner integral.

$$\int \prod_{i=1}^{n} c_i \mathcal{N}\left(\mathbf{y}|\left\langle \mathbf{f}\right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right)p(\mathbf{X})\mathrm{d}\mathbf{X}$$

$$\geq \left\langle \sum_{i=1}^{n} \log c_i \right\rangle_{q(\mathbf{X})} + \left\langle \log \mathcal{N}\left(\mathbf{y}|\left\langle \mathbf{f}\right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right)\right\rangle_{q(\mathbf{X})} + \mathrm{KL}\left(q(\mathbf{X})\,\|\,p(\mathbf{X})\right)$$

- Which is analytically tractable for Gaussian $q(\mathbf{X})$ and some covariance functions.

Required Expectations

- Need expectations under $q(\mathbf{X})$ of:

$$\log c_i = \frac{1}{2\sigma^2}\left[k_{i,i} - \mathbf{k}_{i,\mathbf{u}}^\top \mathbf{K}_{\mathbf{u},\mathbf{u}}^{-1}\mathbf{k}_{i,\mathbf{u}}\right]$$

and

$$\log \mathcal{N}\left(\mathbf{y}|\left\langle \mathbf{f}\right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{Y})}, \sigma^2\mathbf{I}\right) = -\frac{1}{2}\log 2\pi\sigma^2 - \frac{1}{2\sigma^2}\left(y_i - \mathbf{K}_{\mathbf{f},\mathbf{u}}\mathbf{K}_{\mathbf{u},\mathbf{u}}^{-1}\mathbf{u}\right)^2$$

- This requires the expectations

$$\left\langle \mathbf{K}_{\mathbf{f},\mathbf{u}}\right\rangle_{q(\mathbf{X})}$$

and

$$\left\langle \mathbf{K}_{\mathbf{f},\mathbf{u}}\mathbf{K}_{\mathbf{u},\mathbf{u}}^{-1}\mathbf{K}_{\mathbf{u},\mathbf{f}}\right\rangle_{q(\mathbf{X})}$$

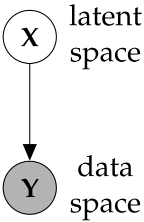
which can be computed analytically for some covariance functions.

Priors for Latent Space

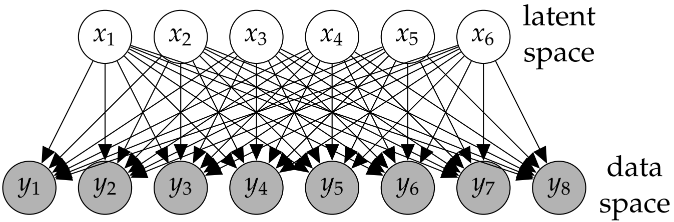
Titsias and Lawrence (2010)

- Variational marginalization of $\mathbf{X}$ allows us to learn parameters of $p(\mathbf{X})$.
- Standard GP-LVM where $\mathbf{X}$ learnt by MAP, this is not possible (see e.g. Wang et al., 2008).
- First example: learn the dimensionality of latent space.

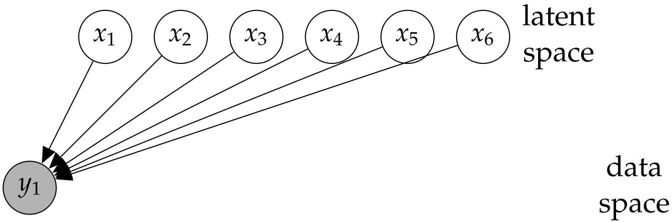
Graphical Representations of GP-LVM



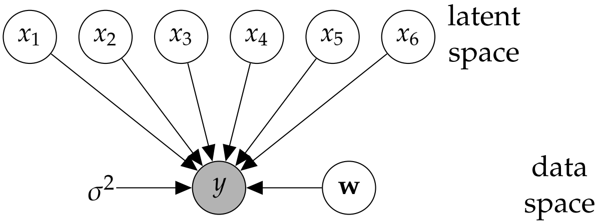
Graphical Representations of GP-LVM



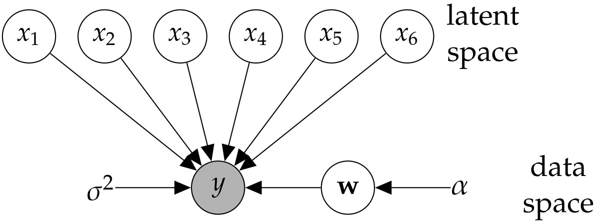
Graphical Representations of GP-LVM



Graphical Representations of GP-LVM



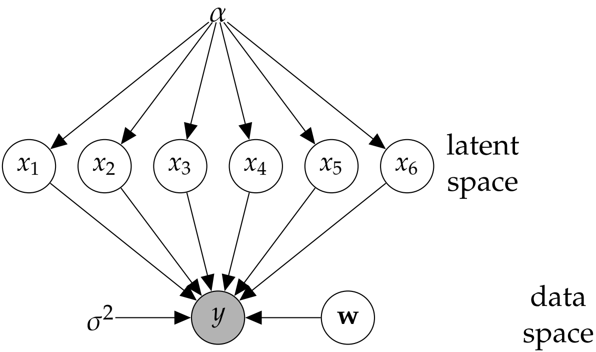
Graphical Representations of GP-LVM



$$\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \alpha \mathbf{I}) \quad \mathbf{x} \sim \mathcal{N}(0, \mathbf{I})$$

$$y \sim \mathcal{N}\left(\mathbf{x}^\top \mathbf{w}, \sigma^2\right)$$

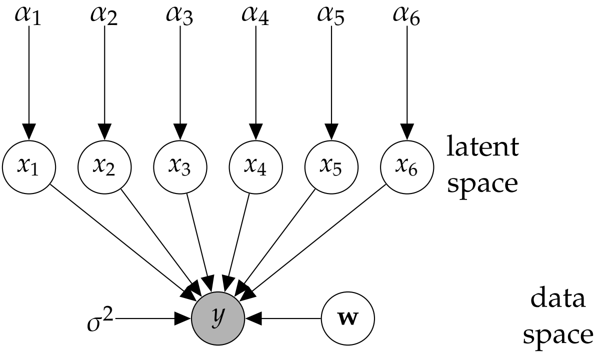
Graphical Representations of GP-LVM



$$\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}) \quad \mathbf{x} \sim \mathcal{N}(0, \alpha\mathbf{I})$$

$$y \sim \mathcal{N}\left(\mathbf{x}^\top \mathbf{w}, \sigma^2\right)$$

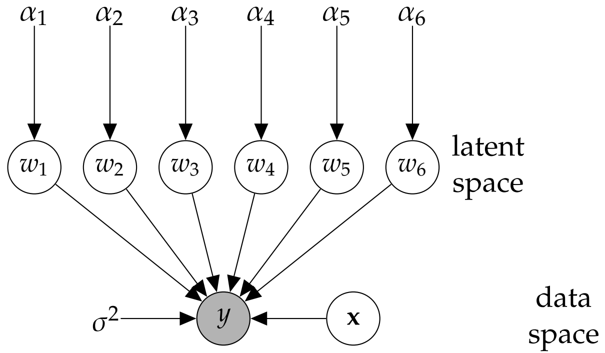
Graphical Representations of GP-LVM



$$\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}) \quad x_i \sim \mathcal{N}(0, \alpha_i)$$

$$y \sim \mathcal{N}\left(\mathbf{x}^\top \mathbf{w}, \sigma^2\right)$$

Graphical Representations of GP-LVM



$$w_i \sim \mathcal{N}(0, \alpha_i) \quad \mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$$

$$y \sim \mathcal{N}\left(\mathbf{x}^\top \mathbf{w}, \sigma^2\right)$$

Non-linear $f(\mathbf{x})$

- In linear case equivalence because $f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x}$

$$p(w_i) \sim \mathcal{N}(\mathbf{0}, \alpha_i)$$

- In non linear case, need to scale columns of $\mathbf{X}$ in prior for $f(\mathbf{x})$.
- This implies scaling columns of $\mathbf{X}$ in covariance function

$$k(\mathbf{x}_{i,:}, \mathbf{x}_{j,:}) = \exp\left(-\frac{1}{2}(\mathbf{x}_{:,i} - \mathbf{x}_{:,j})^\top \mathbf{A}(\mathbf{x}_{:,i} - \mathbf{x}_{:,j})\right)$$

$\mathbf{A}$ is diagonal with elements α_i^2. Now keep prior spherical

$$p(\mathbf{X}) = \prod_{j=1}^{q} \mathcal{N}\left(\mathbf{x}_{:,j}|\mathbf{0}, \mathbf{I}\right)$$

- Covariance functions of this type are known as ARD (see e.g. Neal, 1996; MacKay, 2003; Rasmussen and Williams, 2006).

Other Priors on $\mathbf{X}$

- Dynamical prior gives us Gaussian process dynamical system (Wang et al., 2006; Damianou et al., 2011)
- Structured learning prior gives us (soft) manifold sharing (Shon et al., 2006; Navaratnam et al., 2007; Ek et al., 2008b,a; Damianou et al., 2012)
- Gaussian process prior gives us Deep Gaussian Processes (Lawrence and Moore, 2007; Damianou and Lawrence, 2013)

References I

W. V. Baxter and K.-I. Anjyo. Latent doodle space. In *EUROGRAPHICS*, volume 25, pages 477–485, Vienna, Austria, September 4-8 2006.

A. Damianou, C. H. Ek, M. K. Titsias, and N. D. Lawrence. Manifold relevance determination. In J. Langford and J. Pineau, editors, *Proceedings of the International Conference in Machine Learning*, volume 29, San Francisco, CA, 2012. Morgan Kauffman. [PDF].

A. Damianou and N. D. Lawrence. Deep Gaussian processes. In C. Carvalho and P. Ravikumar, editors, *Proceedings of the Sixteenth International Workshop on Artificial Intelligence and Statistics*, volume 31, pages 207–215, AZ, USA, 4 2013. JMLR W&CP 31. [PDF].

A. Damianou, M. K. Titsias, and N. D. Lawrence. Variational Gaussian process dynamical systems. In P. Bartlett, F. Peirrera, C. K. I. Williams, and J. Lafferty, editors, *Advances in Neural Information Processing Systems*, volume 24, Cambridge, MA, 2011. MIT Press. [PDF].

C. H. Ek, J. Rihan, P. H. S. Torr, G. Rogez, and N. D. Lawrence. Ambiguity modeling in latent spaces. In A. Popescu-Belis and R. Stiefelhagen, editors, *Machine Learning for Multimodal Interaction (MLMI 2008)*, LNCS, pages 62–73. Springer-Verlag, 28–30 June 2008a. [PDF].

C. H. Ek, P. H. S. Torr, and N. D. Lawrence. Gaussian process latent variable models for human pose estimation. In A. Popescu-Belis, S. Renals, and H. Bourlard, editors, *Machine Learning for Multimodal Interaction (MLMI 2007)*, volume 4892 of *LNCS*, pages 132–143, Brno, Czech Republic, 2008b. Springer-Verlag. [PDF].

K. Grochow, S. L. Martin, A. Hertzmann, and Z. Popovic. Style-based inverse kinematics. In *ACM Transactions on Graphics (SIGGRAPH 2004)*, pages 522–531, 2004.

N. D. Lawrence. Gaussian process models for visualisation of high dimensional data. In S. Thrun, L. Saul, and B. Schölkopf, editors, *Advances in Neural Information Processing Systems*, volume 16, pages 329–336, Cambridge, MA, 2004. MIT Press.

N. D. Lawrence. Probabilistic non-linear principal component analysis with Gaussian process latent variable models. *Journal of Machine Learning Research*, 6:1783–1816, 11 2005.

N. D. Lawrence and A. J. Moore. Hierarchical Gaussian process latent variable models. In Z. Ghahramani, editor, *Proceedings of the International Conference in Machine Learning*, volume 24, pages 481–488. Omnipress, 2007. [Google Books] . [PDF].

References II

D. J. C. MacKay. *Information Theory, Inference and Learning Algorithms*. Cambridge University Press, Cambridge, U.K., 2003. [Google Books] .

R. Navaratnam, A. Fitzgibbon, and R. Cipolla. The joint manifold model for semi-supervised multi-valued regression. In *IEEE International Conference on Computer Vision (ICCV)*. IEEE Computer Society Press, 2007.

R. M. Neal. *Bayesian Learning for Neural Networks*. Springer, 1996. Lecture Notes in Statistics 118.

V. Priacuriu and I. D. Reid. Nonlinear shape manifolds as shape priors in level set segmentation and tracking. In *Proceedings of the IEEE Computer Society Conference on Computer Vision and Pattern Recognition*, 2011a.

V. Priacuriu and I. D. Reid. Shared shape spaces. In *IEEE International Conference on Computer Vision (ICCV)*, 2011b.

C. E. Rasmussen and C. K. I. Williams. *Gaussian Processes for Machine Learning*. MIT Press, Cambridge, MA, 2006. [Google Books] .

A. P. Shon, K. Grochow, A. Hertzmann, and R. P. N. Rao. Learning shared latent structure for image synthesis and robotic imitation. In Weiss et al. (2006).

M. E. Tipping and C. M. Bishop. Probabilistic principal component analysis. *Journal of the Royal Statistical Society, B*, 6 (3):611–622, 1999. [PDF]. [DOI].

M. K. Titsias and N. D. Lawrence. Bayesian Gaussian process latent variable model. In Y. W. Teh and D. M. Titterington, editors, *Proceedings of the Thirteenth International Workshop on Artificial Intelligence and Statistics*, volume 9, pages 844–851, Chia Laguna Resort, Sardinia, Italy, 13-16 May 2010. JMLR W&CP 9. [PDF].

R. Urtasun, D. J. Fleet, and P. Fua. 3D people tracking with Gaussian process dynamical models. In *Proceedings of the IEEE Computer Society Conference on Computer Vision and Pattern Recognition*, pages 238–245, New York, U.S.A., 17–22 Jun. 2006. IEEE Computer Society Press.

R. Urtasun, D. J. Fleet, A. Hertzmann, and P. Fua. Priors for people tracking from small training sets. In *IEEE International Conference on Computer Vision (ICCV)*, pages 403–410, Bejing, China, 17–21 Oct. 2005. IEEE Computer Society Press.

J. M. Wang, D. J. Fleet, and A. Hertzmann. Gaussian process dynamical models. In Weiss et al. (2006).

J. M. Wang, D. J. Fleet, and A. Hertzmann. Gaussian process dynamical models for human motion. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 30(2):283–298, 2008. ISSN 0162-8828. [DOI].

Y. Weiss, B. Schölkopf, and J. C. Platt, editors. *Advances in Neural Information Processing Systems*, volume 18, Cambridge, MA, 2006. MIT Press.