Dimensionality Reduction

MLAI: Week 8

Neil D. Lawrence

Department of Computer Science
Sheffield University

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Review

- Last time: Looked at classification.
- Introduced Naive Bayes and Logistic Regression.
- This time: Dimensionality reduction.

Outline

Clustering

Classification

Clustering

- Divide data into discrete groups according to characteristics.
  - For example different animal species.
  - Different political parties.
- Determine the allocation to the groups and (harder) number of different groups.

K-means Clustering

An Algorithm

- *Require*: Set of K cluster centers & assignment of each point to a cluster.
  - Initialize cluster centers as data points.
  - Assign each data point to nearest cluster center.
  - Update each cluster center by setting it to the mean of assigned data points.

Objective Function

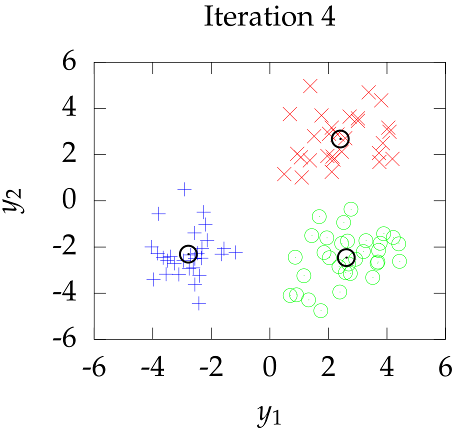
- This minimizes the objective:

$$\sum_{j=1}^{K} \sum_{i \text{ allocated to } j} \left(\mathbf{y}_{i,:} - \boldsymbol{\mu}_{j,:}\right)^\top \left(\mathbf{y}_{i,:} - \boldsymbol{\mu}_{j,:}\right)$$

- i.e. it minimizes the sum of Euclidean squared distances between points and their associated centers.
- The minimum is not guaranteed to be *global* or *unique*.
  - This objective is a non-convex optimization problem.

K-means Clustering

- K-means clustering.
  - Update each center by setting to the mean of the allocated points.



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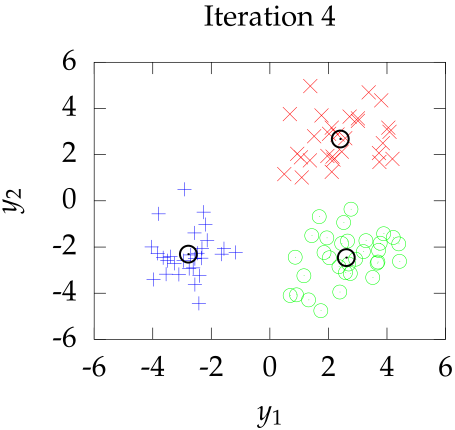

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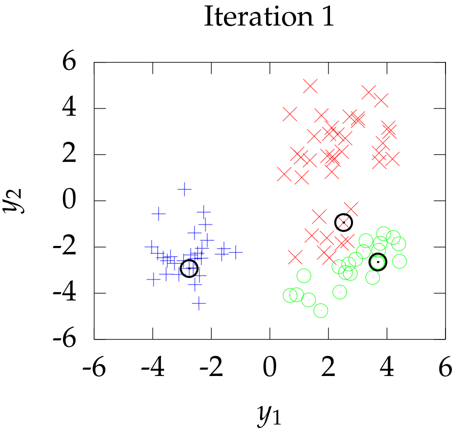

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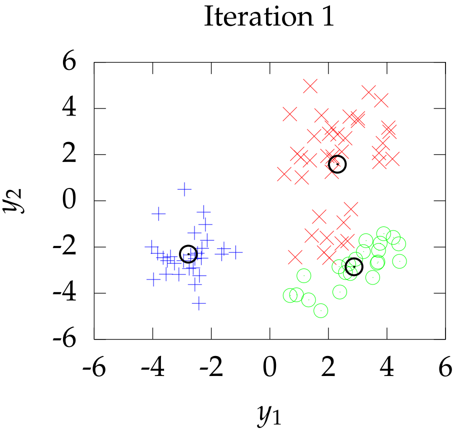

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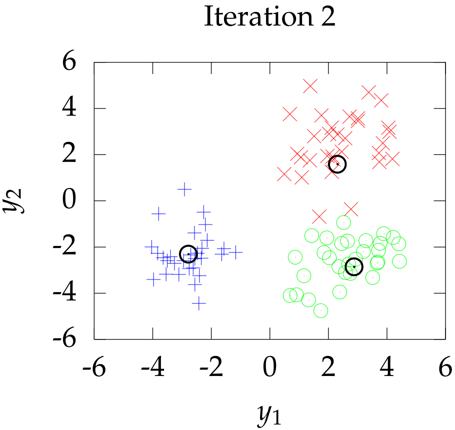

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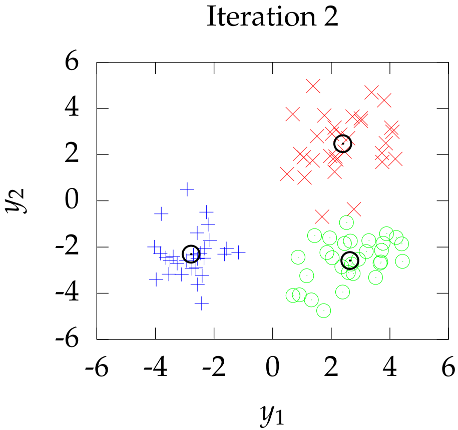

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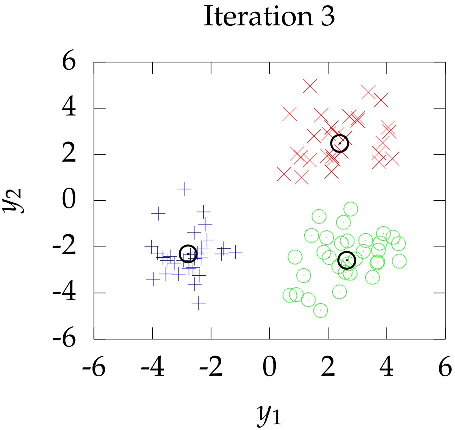

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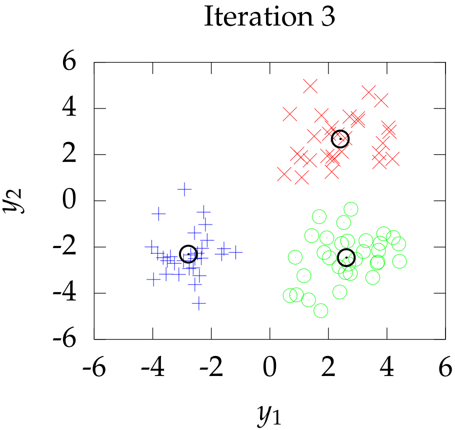

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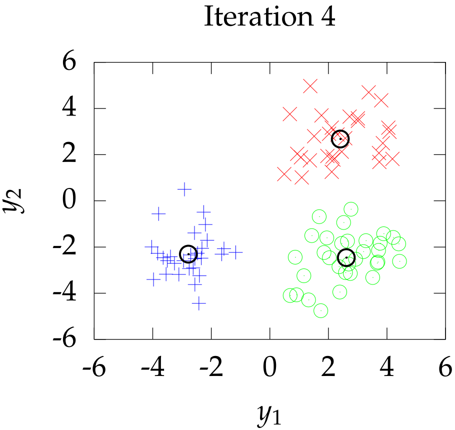

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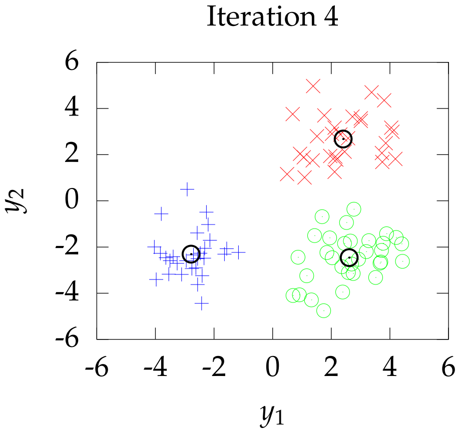
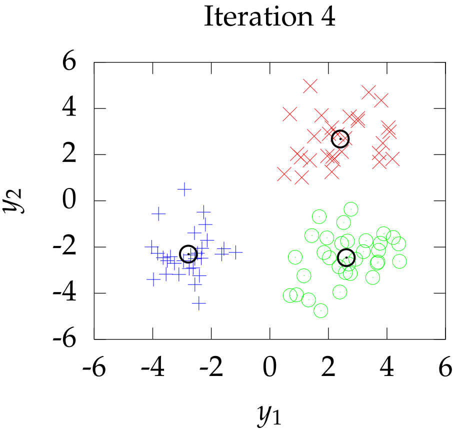


K-means Clustering

- *K*-means clustering.
  - Allocation doesn't change so stop.



Other Clustering Approaches

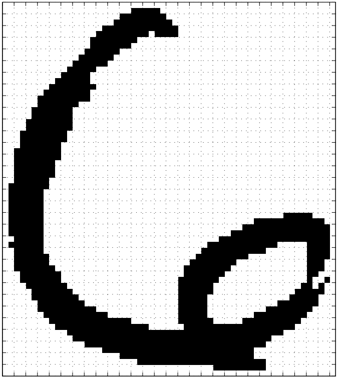
- Spectral clustering (Shi and Malik, 2000; Ng et al., 2002).
  - Allows clusters which aren't convex hulls.
- Dirichlet processes
  - A probabilistic formulation for a clustering algorithm that is non-parameteric.

Outline

High Dimensional Data

USPS Data Set Handwritten Digit

- 3648 Dimensions
- 64 rows by 57 columns



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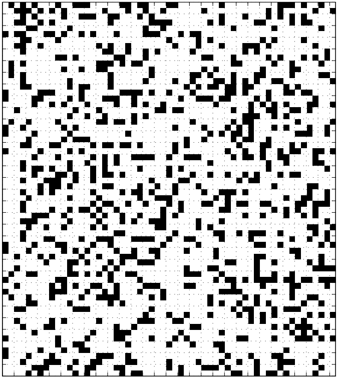

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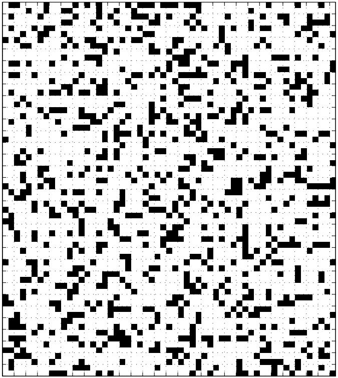

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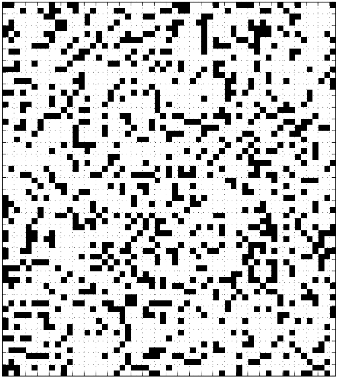

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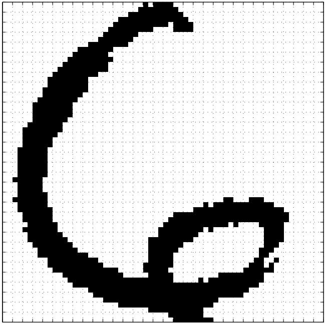
Simple Model of Digit

- Rotate a 'Prototype'



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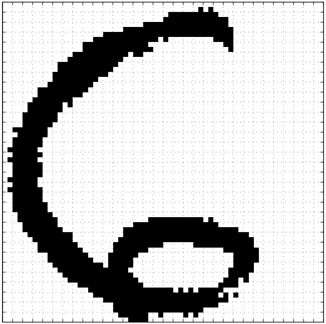

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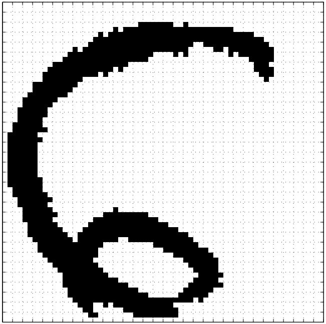

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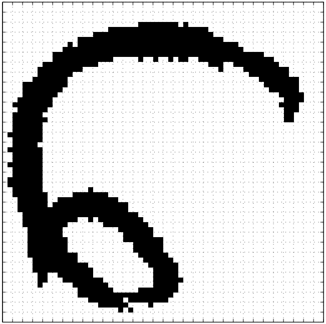

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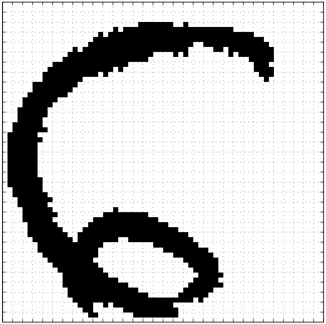

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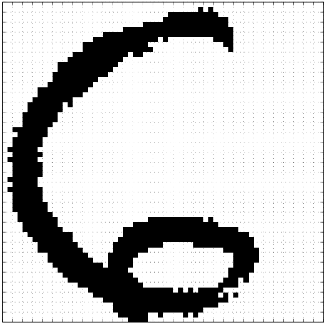

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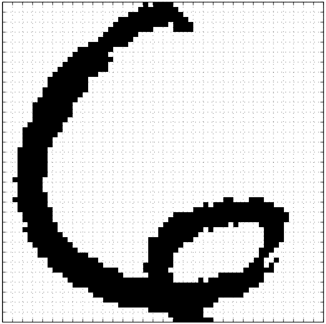

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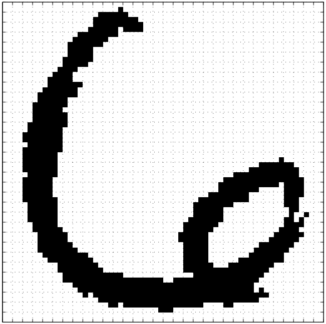

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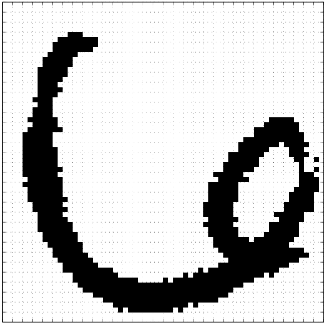

MATLAB Demo

```
demDigitsManifold([1 2], 'all')
```

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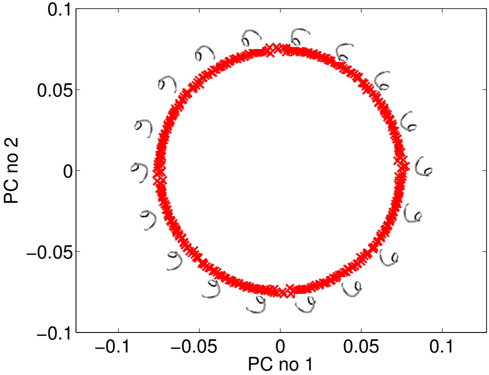

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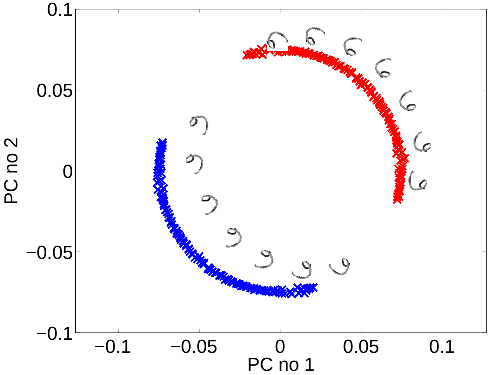
```
demDigitsManifold([1 2], 'sixnine')
```



Pure Rotation is too Simple

- In practice the data may undergo several distortions.
  - *e.g.* digits undergo 'thinning', translation and rotation.
- For data with 'structure':
- we expect fewer distortions than dimensions;
- we therefore expect the data to live on a lower dimensional manifold.
- Conclusion: deal with high dimensional data by looking for lower dimensional non-linear embedding.

Principal Component Analysis

- How do we find these directions?
- Rotate to find directions in data with maximal variance.
  - This is known as PCA (Hotelling, 1933).
- Rotate data to extract directions of maximum variance.
- Do this by diagonalizing the sample covariance matrix

$$\mathbf{S} = n^{-1} \sum_{i=1}^{n} \left(\mathbf{y}_i - \boldsymbol{\mu}\right)\left(\mathbf{y}_i - \boldsymbol{\mu}\right)^{\top}$$

Principal Component Analysis

- Find a direction in the data, $\mathbf{x} = \mathbf{R}\mathbf{y}$, for which variance is maximized.

Lagrangian

- Solution is found via constrained optimisation (which uses *Lagrange* multipliers):

$$L\left(\mathbf{r}_1, \lambda_1\right) = \mathbf{r}_1^\top \mathbf{S}\mathbf{r}_1 + \lambda_1 \left(1 - \mathbf{r}_1^\top \mathbf{r}_1\right)$$

- Gradient with respect to $\mathbf{r}_1$

$$\frac{\mathrm{d}L\left(\mathbf{r}_1, \lambda_1\right)}{\mathrm{d}\mathbf{r}_1} = 2\mathbf{S}\mathbf{r}_1 - 2\lambda_1\mathbf{r}_1$$

  rearrange to form

$$\mathbf{S}\mathbf{r}_1 = \lambda_1\mathbf{r}_1.$$

  Which is known as an *eigenvalue* problem.
- Further directions can also be shown to be eigenvectors of the covariance.

Linear Dimensionality Reduction

Linear Latent Variable Model

- Represent data, $\mathbf{Y}$, with a lower dimensional set of latent variables $\mathbf{X}$.
- Assume a linear relationship of the form

$$\mathbf{y}_{i,:} = \mathbf{W}\mathbf{x}_{i,:} + \boldsymbol{\epsilon}_{i,:},$$

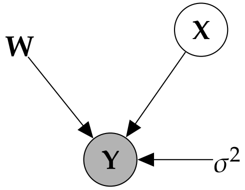
  where

$$\boldsymbol{\epsilon}_{i,:} \sim \mathcal{N}\left(\mathbf{0}, \sigma^2\mathbf{I}\right).$$

Linear Latent Variable Model

Probabilistic PCA

- Define *linear-Gaussian relationship* between latent variables and data.



$$p\left(\mathbf{Y}|\mathbf{X},\mathbf{W}\right) = \prod_{i=1}^{n} \mathcal{N}\left(\mathbf{y}_{i,:}|\mathbf{W}\mathbf{x}_{i,:},\sigma^2\mathbf{I}\right)$$

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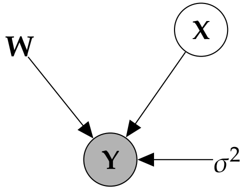
- **Standard** Latent variable approach:



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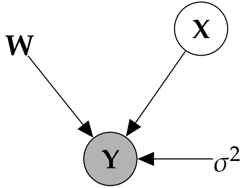

$$p(\mathbf{Y}|\mathbf{X},\mathbf{W}) = \prod_{i=1}^{n} \mathcal{N}\left(\mathbf{y}_{i,:}|\mathbf{W}\mathbf{x}_{i,:},\sigma^2\mathbf{I}\right)$$

$$p(\mathbf{X}) = \prod_{i=1}^{n} \mathcal{N}\left(\mathbf{x}_{i,:}|\mathbf{0},\mathbf{I}\right)$$

Linear Latent Variable Model

Probabilistic PCA

- Define *linear-Gaussian relationship* between latent variables and data.
- **Standard** Latent variable approach:
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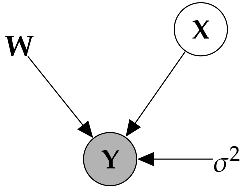
  - Integrate out *latent variables*.



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$$p\left(\mathbf{Y}|\mathbf{W}\right)=\prod_{i=1}^{n}\mathcal{N}\left(\mathbf{y}_{i,:}|\mathbf{0},\mathbf{W}\mathbf{W}^{\top}+\sigma^2\mathbf{I}\right)$$

Computation of the Marginal Likelihood

$$\mathbf{y}_{i,:} = \mathbf{W}\mathbf{x}_{i,:} + \boldsymbol{\epsilon}_{i,:}, \quad \mathbf{x}_{i,:} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{I}\right), \quad \boldsymbol{\epsilon}_{i,:} \sim \mathcal{N}\left(\mathbf{0}, \sigma^2\mathbf{I}\right)$$

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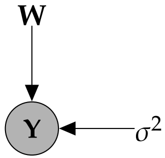
Probabilistic PCA Max. Likelihood Soln (Tipping and Bishop, 1999)



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$$\log p(\mathbf{Y}|\mathbf{W}) = -\frac{n}{2}\log|\mathbf{C}| - \frac{1}{2}\mathrm{tr}\left(\mathbf{C}^{-1}\mathbf{Y}^{\top}\mathbf{Y}\right) + \text{const.}$$

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$$\mathbf{W} = \mathbf{U}_q\mathbf{L}\mathbf{R}^{\top}, \quad \mathbf{L} = \left(\mathbf{\Lambda}_q - \sigma^{2}\mathbf{I}\right)^{\frac{1}{2}}$$

where $\mathbf{R}$ is an arbitrary rotation matrix.

Reading

- Chapter 7 of Rogers and Girolami up to pg 249.

References I

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